ON A METHOD OF SOLVING THE ELASTO-PLASTIC PROBLEM

(OB ODNOM METODE RESHENIIA UPRUGO-PLASTICHESKOI ZADACHI)

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Let there be given, on an unknown contour L in the plane of the complex variable $z = x + iy$, the second derivatives of a biharmonic function which are known functions of the coordinates x and y. It is required to determine the boundary L and the biharmonic function. The elasto-plastic problem for a body in a state of plane strain or plane stress can be reduced to this mathematical formulation in the case when the plastic zone completely surrounds the contour of the body, because the stresses in the plastic region are as a rule determined directly by the boundary loads [1]. Some problems of plate buckling and of fracture of materials may be reduced to an analogous mathematical problem [2].

In the case when the given boundary functions are the corresponding second derivatives of a biharmonic function, the problem may be solved by the method of Galin [3].

Below, a method of solution of a certain class of the indicated problems is advanced, in which the boundary functions do not have to satisfy the last condition. The problem of elasto-plastic equilibrium of plates with a circular hole is considered in detail in the case when the contour of the cavity is subjected to a constant normal loading and the tangential loading is zero, while at infinity a homogeneous state of stress is assumed.

1. Formulation of problems. Let an infinite elasto-plastic body, in a state of plane strain or plane stress, possess a cavity subjected to an arbitrary normal and tangential loading. At infinity stresses exist, which are polynomial functions of Cartesian coordinates x and y. The origin of the coordinates is located within the cavity.

The whole cavity is completely within the plastic zone. We assume that the stresses in the plastic region are determined solely by the contour shape of the cavity and by the boundary loading and do not depend on the state of stress in the elastic region. Then the stress may be found from the solution of the equations of plasticity theory [1]. We shall assume them to be known functions x and y.

In the plane problem of the theory of elasticity the components of the tensor of stresses σ_x, σ_y, τ_{xy} are determined by the formulas of Kolosov-Muskhelishvili [4]

$$\sigma_x + \sigma_y = 2\,[\Phi(z) + \overline{\Phi(z)}], \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = 2\,[\bar{z}\,\Phi'(z) + \Psi(z)] \quad (1.1)$$

Here $\Phi(z)$ and $\Psi(z)$ are analytical functions of $z = x + iy$, which behave at infinity as

$$\Phi(z) = d_n z^n + \ldots + d_0 + O(z^{-1}), \qquad \Psi(z) = e_n z^n + \ldots + e_0 + O(z^{-1}) \quad (1.2)$$

We recall that the stresses σ_x, σ_y, τ_{xy} are expressed by means of the second derivatives of a biharmonic function $V(x, y)$ as:

$$\sigma_x = V_{yy}, \qquad \sigma_y = V_{xx}, \qquad \tau_{xy} = -V_{xy}$$

Along the unknown contour L, which separates the elastic from the plastic region, all the stresses are continuous. From this, by formulas (1.1), we obtain the following boundary value problem for the exterior of the contour L

$$\Phi(z) + \overline{\Phi(z)} = f_1(x, y) \quad \text{on } L, \qquad \bar{z}\,\Phi'(z) + \Psi(z) = f_2(x, y) \quad \text{on } L \quad (1.3)$$

Here $f_1(x, y)$ and $f_2(x, y)$ are continuous functions, known from the solution of the corresponding problem of the theory of plasticity (the first of these is real, the second - complex). It is required to find the contour L and functions $\Phi(z)$ and $\Psi(z)$ using the boundary conditions (1.2) and (1.3).

In completely analogous fashion one formulates the interior plane elasto-plastic problem, when the elastic region occupies the interior of the contour L (thereby the requirement in condition (1.2) is, naturally, omitted). In the sequel, for the sake of brevity, we consider only the exterior elasto-plastic problem.

The mathematical formulation of the boundary value problem (1.3), (1.2) may be also invoked in describing physical problems of a somewhat different type. Let it be assumed that an infinite plate with a cavity, subjected to loads, is extended at infinity and, being in a state of plane stress, a zone of local buckling appears, which surrounds the whole contour of the cavity. The buckling may occur also in the case when the cavity is absent, but the plate is subjected to intensive body forces. Let us assume that, based on certain considerations, we know the state of stress in the buckling zone (even within an accuracy with several undetermined constants). In particular, the stresses in the buckling zone are easily determined in the case of plates with vanishing bending rigidity [2]. Then, on the unknown contour of the buckling zone, we obviously obtain the boundary problem (1.3).

If the contour of the body in a state of plane strain or plane stress is completely surrounded by a certain region of limiting equilibrium (zone of fracture), such that in the region of limiting equilibrium the stresses may be determined by solving a corresponding statically determinate problem which does not depend on the state of stress in the remaining region, then the corresponding boundary value problem coincides with the limiting problem (1.3).

In solving elasto-plastic problems we shall adopt in the sequel the usual assumptions [5]: 1) each characteristic, emanating from the contour of the body, intersects the unknown contour L at one point, 2) during loading the contours of separation of elastic and plastic zones successively contain each other.

2. Method of solution of the boundary value problem.

1. We pass to the parametric plane of the complex variable ζ with the aid of the transformation $z = \omega(\zeta)$. The analytic function $\omega(\zeta)$ maps conformally the exterior of a unit circle of the plane ζ on the exterior of the unknown contour L of the plane z with the correspondence of infinite points $\omega(\infty) = \infty$; it has to be determined during the process of solving the problem. We introduce the notation

$$\varphi(\zeta) = \Phi[\omega(\zeta)], \quad \psi(\zeta) = \Psi[\omega(\zeta)], \quad F_k[\omega(\zeta), \overline{\omega(\zeta)}] = \tag{2.1}$$
$$= f_k\{\tfrac{1}{2}[\omega(\zeta) + \overline{\omega(\zeta)}], \tfrac{1}{2i}[\omega(\zeta) - \overline{\omega(\zeta)}]\} \quad (k = 1,2)$$

In the notation adopted we obtain on the plane ζ the following boundary value problem from the boundary conditions (1.3) for determination of three analytic functions $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$

$$\varphi(\zeta) + \overline{\varphi(\zeta)} = F_1[\omega(\zeta), \overline{\omega(\zeta)}] \quad \text{for } |\zeta| = 1 \tag{2.2}$$
$$\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\varphi'(\zeta) + \psi(\zeta) = F_2[\omega(\zeta), \overline{\omega(\zeta)}] \quad \text{for } |\zeta| = 1$$

The three functions $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$ behave at infinity, because of (1.2), as follows:

$$\varphi(\zeta) = O(\zeta^n), \qquad \psi(\zeta) = O(\zeta^n), \qquad \omega(\zeta) = O(\zeta) \tag{2.3}$$

We consider the second boundary condition (2.2). It represents a certain finite equation with respect to $\overline{\omega(\zeta)}$.

Proposition 2.1. Let the second boundary condition (2.2) be solved with respect to $\overline{\omega(\zeta)}$

$$\overline{\omega(\zeta)} = \Gamma[\omega(\zeta), \omega'(\zeta), \varphi'(\zeta), \psi(\zeta)] \tag{2.4}$$

and the function $\chi(\zeta)$

$$\chi(\zeta) = \Gamma[\omega(\zeta), \omega'(\zeta), \varphi'(\zeta), \psi(\zeta)] \tag{2.5}$$

is analytic in the exterior of the unit circle $|\zeta| > 1$, except, possibly, at a finite number of isolated singular points of single-valued character. Then the boundary value problem (2.2) can be solved in closed form.

Indeed, let the functions $\omega(\zeta)$ and $\chi(\zeta)$ as $\zeta \to \infty$ be of the form

$$\omega(\zeta) = c_0\zeta + \sum_{k=1}^{\infty} c_k\zeta^{-k}, \qquad \chi(\zeta) - \chi_0(\zeta) = O(\zeta^\nu) \qquad (\nu \geqslant 0) \tag{2.6}$$

Here $\chi_0(\zeta)$ is a function analytic in the plane ζ, except at singular points of the function $\chi(\zeta)$, at which it possesses singularities, which coincide with the singularities of the function $\chi(\zeta)$; the function $\chi_0(\zeta)$ is known to an accuracy within, possibly, undetermined constants. If the function $\chi(\zeta)$ possesses an essential singular point at infinity, then the corresponding singularity is also included in the expression for $\chi_0(\zeta)$. The solution of the boundary value problem (2.4), obviously, is written in the form

$$\omega(\zeta) = c_0\zeta + \overline{P}_\nu(\zeta^{-1}) + \overline{\chi}_0(\zeta^{-1}), \qquad \chi(\zeta) = \overline{c}_0\zeta^{-1} + P_\nu(\zeta) + \chi_0(\zeta) \tag{2.7}$$

Here P_ν is a polynomial of power ν with as yet undetermined coefficients. Subsequently, the function $\varphi(\zeta)$ is found from the solution of the Dirichlet problem (2.2). All unknown constants are determined from finite equations, obtained by expanding the found functions $\varphi(\zeta)$, $\omega(\zeta)$ and $\psi(\zeta)$ in the vicinity of singular points of the function $\chi(\zeta)$ and at infinity, using equation (2.5).

2. In the proposition advanced above, the analyticity condition of the function $\chi(\zeta)$ in the region exterior to the unit circle is an *a posteriori* one. Let us indicate a procedure which is convenient in

practical applications of the proposition mentioned.

Let us consider the following equation

$$\frac{\bar{\omega}(\zeta^{-1})}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta) = F_2[\omega(\zeta),\ \bar{\bar{\omega}}(\zeta^{-1})] \qquad (|\zeta| > 1) \qquad (2.8)$$

The function $\bar{\omega}(\zeta^{-1})$ is obviously analytic within the unit circle $|\zeta| < 1$, except at the origin of the coordinates, at which it possesses a pole of first order.

Proposition 2.2. Let the function $\omega(\zeta)$ be analytic in the whole plane ζ, except at the infinite point, at which it possesses a pole of first order and, possibly, a finite number of isolated singular points of single-valued character, placed within the unit circle $|\zeta| < 1$.

Then the second boundary condition (2.2) may be analytically continued into the exterior of the unit circle $|\zeta| > 1$ with the aid of the functional equation (2.8).

Indeed, under the assumptions stated, the function $\omega(\zeta)$ may be written in the form (2.7). In particular, as is the case in the example considered below, the function $\chi_0(\zeta)$ may be equal to zero identically. Then, obviously

$$\bar{\omega}(\zeta^{-1}) = \bar{c}_0\zeta^{-1} + P_\nu(\zeta) + \chi_0(\zeta) \qquad (2.9)$$

and the left part of the functional equation (2.8) is an analytic function in the exterior of the unit circle, except at the infinite point and, possibly, at a finite number of singular points of single-valued character. Thus, also the right part of the functional equation (2.8), i.e. the function $F_2[\omega(\zeta),\ \bar{\omega}(\zeta^{-1})]$, must be analytic in the exterior of the unit circle, except at the infinite point and, possibly, at a finite number of singular points of single-valued character, whose singularities coincide with the corresponding singularities of the left part. Thereby, the second boundary condition (2.2) will obviously be satisfied.

Thus, the necessary feature that the function $\omega(\zeta)$ has the form (2.7), is the analyticity of the right part of the functional equation (2.8) in the exterior of the unit circle $|\zeta| > 1$, except a finite number of singular points of single-valued character. This feature will also be sufficient if the singularities of the left and the right part of the functional equation (2.8) can be chosen such that they coincide.

This feature permits one sometimes to find quite easily the closed solutions of the boundary value problem (2.2) also in that case, when it is not known whether the function $F_2[\omega(\zeta), \bar{\omega}(\zeta^{-1})]$ for $|\zeta| > 1$ is analytic or not. To this end one should formally substitute expressions (2.7) and (2.9) for the functions $\omega(\zeta)$ and $\bar{\omega}(\zeta^{-1})$ into the functional equation (2.8), require the analyticity of the function $F_2[\omega(\zeta), \bar{\omega}(\zeta^{-1})]$ almost everywhere in $|\zeta| > 1$ and equate the singularities of the left and the right part of the functional equation (2.8).

If it is possible to satisfy these conditions by a suitable selection of the undetermined coefficients and functions, then the existence of the solution (2.7) is proved.

Note. The indicated argumentation is valid, obviously, with unessential changes not only for the second boundary condition (2.2), but also for a boundary condition of a more general form

$$f\,[\overline{\omega(\zeta)},\ a_1(\zeta),\ a_2(\zeta), \ldots] = 0 \quad \text{for} \quad |\zeta| = 1$$

where $a_i(\zeta)$ are analytic functions, f is a differentiable function, in particular for the problem of elasto-plastic torsion, when the elastic kernel is completely surrounded by the plastic zone.

3. Plate with a circular hole. 1. Let an infinite plate in a plane state of stress possess a circular hole of radius R with center at the origin of the coordinate system. A normal loading $\sigma_r = p$ is applied to the contour of the hole, and the tangential loading equals zero $\tau_{r\theta} = 0$ (r, θ are polar coordinates). At infinity the homogeneous state of stress exists

$$\sigma_x = \sigma_x^{\infty}, \qquad \sigma_y = \sigma_y^{\infty}, \qquad \tau_{xy} = 0 \tag{3.1}$$

Let us assume that the plastic region surrounds completely the circular hole. We take the Tresca-St. Venant condition as the plasticity condition in the plastic region. Let us assume that in the plastic zone the inequality $\sigma_\theta \geqslant \sigma_r > 0$ is valid. The characteristics in the plastic zone will be radial straight lines and the stresses are equal [6]

$$\sigma_\theta = \sigma_s, \qquad \sigma_r = \sigma_s + (p - \sigma_s) R / r, \qquad \tau_{r\theta} = 0 \tag{3.2}$$

(σ_s is the plasticity constant)

To satisfy the inequality $\sigma_\theta \geqslant \sigma_r > 0$, the loading p, obviously, must satisfy the condition $p \leqslant \sigma_s$. In the elastic region the fundamental relations of Kolosov-Muskhelishvili [4] are valid

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\,\Phi(z), \quad \sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2e^{2i\theta}\,[\bar{z}\Phi'(z) + \Psi(z)] \tag{3.3}$$

Using (3.2) and (3.3), the boundary conditions on the unknown contour L, which separates the elastic from the plastic region, can be represented as:

$$4\,\mathrm{Re}\,\Phi(z) = 2\sigma_s + \frac{R(p-\sigma_s)}{r}, \quad \bar{z}\,\Phi'(z) + \Psi(z) = \frac{R(\sigma_s - p)}{2r}e^{-2i\theta} \tag{3.4}$$

On the basis of (3.4) the boundary value problem in the plane ζ for the determination of analytic functions $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$ can be written down in the form

$$\begin{aligned} &4\,\mathrm{Re}\,\varphi(\zeta) = 2\sigma_s + (p-\sigma_s)\frac{R}{|\omega(\zeta)|} \qquad \text{for } |\zeta| = 1 \\ &\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta) = \frac{R(\sigma_s - p)\,\overline{\omega(\zeta)}}{2|\omega(\zeta)|\,\omega(\zeta)} \qquad \text{for } |\zeta| = 1 \end{aligned} \tag{3.5}$$

On the basis of formula (3.1) the functions $\varphi(\zeta)$ and $\psi(\zeta)$ behave at infinity as

$$\varphi(\zeta) = \tfrac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + O(\zeta^{-2}), \quad \psi(\zeta) = \tfrac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + O(\zeta^{-2}), \quad \omega(\zeta) = O(\zeta) \tag{3.6}$$

2. Let us consider the functional equation

$$\frac{\varphi'(\zeta)}{\omega'(\zeta)}\bar{\omega}\left(\frac{1}{\zeta}\right) + \psi(\zeta) = \frac{R(\sigma_s - p)\,\bar{\omega}(\zeta^{-1})}{2\omega(\zeta)\sqrt{\omega(\zeta)\,\bar{\omega}(\zeta^{-1})}} \qquad (|\zeta| > 1) \tag{3.7}$$

We seek the solution of the functional equation (3.7) in the form

$$\omega(\zeta) = c_0\zeta + \bar{P}_\nu(\zeta^{-1}) \tag{3.8}$$

Substituting (3.8) into (3.7), and expanding all functions into a series in the vicinity of the point at infinity, it is not difficult to note that $\nu = 3$, such that, if the function $\omega(\zeta)$ is of the form (3.8), then it necessarily is equal to

$$\omega(\zeta) = c_0\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \frac{c_3}{\zeta^3} \tag{3.9}$$

Here $c_0, \ldots, c_3$ are as yet undetermined constants. From conditions of symmetry it follows that the constant c_2 is equal to zero and the remaining constants are real. Let us show that they may be selected in such a way that the right part of the functional equation (3.7) is analytic in the exterior of the unit circle. The function $\omega(\zeta)$ has, obviously, four zeros, located within the unit circle $|\zeta| < 1$, and the function $\bar{\omega}(\zeta^{-1})$ has four zeros located outside the unit circle. In order that the right part of the functional equation (3.7) be analytic in the exterior of the unit circle, it is necessary and sufficient to require pairwise coincidence of zeros of the function $\bar{\omega}(\zeta^{-1})$. Since the function $\bar{\omega}(\zeta^{-1})$ is equal to

$$\bar{\omega}(\zeta^{-1}) = \zeta^{-1}(c_0 + c_1\zeta^2 + c_3\zeta^4) \tag{3.10}$$

then the pairwise coincidence of zeros it is sufficient to require the vanishing of the discriminant of the biquadratic equation

$$c_1^2 = 4c_0c_3 \tag{3.11}$$

Using condition (3.11), the functions $\omega(\zeta)$ and $\bar{\omega}(\zeta^{-1})$ may be written down in the form

$$\omega(\zeta) = \frac{c_0}{\zeta^3}\left(\zeta^2 + \frac{c_1}{2c_0}\right)^2, \qquad \bar{\omega}\left(\frac{1}{\zeta}\right) = \frac{c_3}{\zeta}\left(\zeta^2 + \frac{c_1}{2c_3}\right)^2 \tag{3.12}$$

From the functional equation (3.7) we find

$$\psi(\zeta) = \frac{R(\sigma_s - p)\sqrt{c_3}(\zeta^2 + c_1/2c_3)\zeta^4}{2c_0^{3/2}(\zeta^2 + c_1/2c_0)^3} - \frac{\varphi'(\zeta)}{\omega'(\zeta)}\bar{\omega}(\zeta^{-1}) \tag{3.13}$$

3. Let us find the function $\varphi(\zeta)$. The first boundary condition (3.5), using formulas (3.12), are conveniently written down in the form

$$4\,\mathrm{Re}\,\varphi(\zeta) = 2\sigma_s + \frac{R(p - \sigma_s)\zeta^2}{\sqrt{c_0c_3}(\zeta^2 + c_1/2c_3)(\zeta^2 + c_1/2c_0)} =$$
$$= 2\sigma_s + \frac{2R(p - \sigma_s)\sqrt{c_0c_3}}{c_1(c_3 - c_0)}\left[\frac{1}{\zeta^2 + c_1/2c_3} - \frac{1}{\zeta^2 + c_1/2c_0}\right] \quad \text{for } |\zeta| = 1 \tag{3.14}$$

The function $F^+(\zeta)$ is equal to

$$F^+(\zeta) = -2\bar{\varphi}(\zeta^{-1}) + \frac{2R\sqrt{c_0c_3}(p - \sigma_s)}{c_1(c_3 - c_0)}\frac{\zeta^2}{\zeta^2 + c_1/2c_3} \tag{3.15}$$

and is analytic everywhere within the unit circle $|\zeta| < 1$, while the function

$$F^-(\zeta) = 2\varphi(\zeta) - 2\sigma_s + \frac{2R(p - \sigma_s)\sqrt{c_0c_3}}{c_1(c_3 - c_0)}\frac{\zeta^2}{\zeta^2 + c_1/2c_0} \tag{3.16}$$

is analytic everywhere outside the unit circle $|\zeta| > 1$.

The boundary condition (3.14) may be written down in the form

$$F^+(\zeta) = F^-(\zeta) \quad \text{for } |\zeta| = 1 \tag{3.17}$$

Consequently, the functions $F^+(\zeta)$ and $F^-(\zeta)$ are an analytic continuation of each other across the unit circle. By Liouville's theorem they are identically equal to one and the same constant. From this, and also from the condition at infinity (3.6) for $\varphi(\zeta)$, one easily obtains

$$\varphi(\zeta) = \sigma_s - \frac{1}{4}(\sigma_y^\infty + \sigma_x^\infty) - \frac{R(p-\sigma_s)\sqrt{c_0 c_3}}{c_1(c_3 - c_0)} \frac{\zeta^2}{\zeta^2 + c_1/2c_0} \tag{3.18}$$

where the three as yet undetermined coefficients are related by

$$\sigma_x^\infty + \sigma_y^\infty - 2\sigma_s + \frac{2R(p-\sigma_s)\sqrt{c_0 c_3}}{c_1(c_3 - c_0)} = 0 \tag{3.19}$$

From formula (3.18) we find

$$\varphi'(\zeta) = -\sqrt{\frac{c_3}{c_0}} \frac{R(p-\sigma_s)}{(c_3 - c_0)\zeta^3} + O(\zeta^{-4}) \quad \text{for } \zeta \to \infty \tag{3.20}$$

Using conditions at infinity (3.6) and (3.20), by formula (3.13), we obtain one other relation between the coefficients c_0, c_1, c_3

$$\sigma_y^\infty - \sigma_x^\infty = \frac{R(\sigma_s - p)\sqrt{c_3}}{c_0\sqrt{c_0}} \frac{c_0 + c_3}{c_0 - c_3} \tag{3.21}$$

Thus, the functions $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$ are determined by formulas (3.12), (3.13) and (3.18), while the constants c_0, c_1, c_3 are found from the solution of a system of three finite equations (3.11), (3.19) and (3.21). It is easily verified that all the boundary conditions and the conditions at infinity are satisfied.

The solution of the system (3.11), (3.19) and (3.21) may be represented in the form

$$c_0 = \frac{4R}{\alpha(a^2-4)}, \quad c_1 = \frac{4aR}{\alpha(a^2-4)}, \quad c_3 = \frac{a^2 R}{\alpha(a^2-4)}, \quad \alpha = \frac{\sigma_x^\infty + \sigma_y^\infty - 2\sigma_s}{\sigma_s - p} \tag{3.22}$$

where a is a real root of the cubic equation

$$a^3 + 4a + \beta = 0, \qquad \beta = \frac{8(\sigma_y^\infty - \sigma_x^\infty)}{\sigma_y^\infty + \sigma_x^\infty - 2\sigma_s} \tag{3.23}$$

Since the discriminant $D = -4^4 - 27\beta^2$ of the cubic equation (3.23) is always negative, equation (3.23) has one real and two conjugate complex roots [7].

Finally, the original functions $\omega(\zeta)$, $\varphi(\zeta)$ and $\psi(\zeta)$ in accordance to formulas (3.12), (3.13), (3.18) and (3.22) take on the form

$$\omega(\zeta) = \frac{R(2\zeta^2 + a)^2}{\alpha(a^2 - 4)\zeta^3}, \qquad \varphi(\zeta) = \sigma_s - \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) - \frac{\alpha(p - \sigma_s)\zeta^2}{2\zeta^2 + a}$$

$$\psi(\zeta) = \alpha(p - \sigma_s)\zeta^4 \frac{(a\zeta^2 + a)[2(a^2 + 4)\zeta^2 - a(4 - 3a^2)]}{2(2\zeta^2 + a)^2(2\zeta^2 + a)(2\zeta^2 - 3a)} \tag{3.24}$$

4. Let us investigate the solution obtained (3.24). The equation of the contour L, which separates the elastic and the plastic region, is represented in parametric form by

$$x^*(t) = 4(1 + a)\cos t + a^2 \cos 3t, \quad y^*(t) = 4(1 - a)\sin t - a^2 \sin 3t$$

$$[Rx^*(t) = \alpha(a^2 - 4)x(t), \ Ry^*(t) = \alpha(a^2 - 4)y(t), \ (0 \leqslant t \leqslant 2\pi)] \tag{3.25}$$

Let us find the boundaries of existence of the solution. For the sake of determinateness we assume $\sigma_y^\infty \geqslant \sigma_x^\infty \geqslant 0$. Furthermore, the obvious inequality must be satisfied $\sigma_y^\infty \leqslant \sigma_s$. From here we obtain the inequalities $\alpha < 0$, $\beta < 0$, $a > 0$. In order that the plastic zone surrounds completely the hole, it is necessary that the parameter a satisfies one more inequality

$$a < 2\frac{1 + \alpha}{1 - \alpha} \tag{3.26}$$

Finally, we obtain the following inequalities which determine the boundaries of existence of the solution (3.24)

$$0 > \alpha > -1, \qquad 2\frac{1 + \alpha}{1 - \alpha} > a > 0 \qquad (2 > a > 0) \tag{3.27}$$

5. The derivative of expression (3.24) for $\omega(\zeta)$ will be

$$\omega'(\zeta) = \frac{R(2\zeta^2 + a)(2\zeta^2 - 3a)}{\alpha(a^2 - 4)\zeta^4} \tag{3.28}$$

For conformal mapping, produced by the analytic function $\omega(\zeta)$, it is necessary that everywhere outside of the unit circle its derivative be different from zero. In the opposite case on the contour L, which separates the elastic from the plastic region, a loop of nonsingle-valuedness occurs, which does not possess a physical meaning. To satisfy this condition, the parameter a in accordance with (3.28) must satisfy the inequality

$$0 < a < {}^2/_3 \tag{3.29}$$

From here and from inequality (3.27) it follows that the parameter α should be within the range

$$-1<\alpha<-{}^1/_2 \qquad (3.30)$$

The nonexistence of a physically real solution for values of the parameter α in the range $1/2 < \alpha < 0$ leads one to conclude that the solution of the original elasto-plastic problem, which is continuous in stresses both in the plastic and in the elastic regions, for these values in the parameter α does not exist; one is forced to either introduce stress discontinuities in the plastic region, or adopt a more complete model of the elasto-plastic body. Apparently, in general, the solution of the elasto-plastic problem exists only until the return point appears on the separation contour L. An analogous circumstance occurs in the theory of filtration [8,9], in crack theory [10], and in buckling of plates in membrane formulation [2].

We note that for $\sigma_s = 0$ the solution obtained may be interpreted as a solution of the problem on buckling of a membrane with a circular hole, which is subjected to constant normal tensile stresses at the contour, while the shear stresses on the contour are vanishing.

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